Math 2050, note on lim-sup

1. Bolzano-Weiestrass Theorem

By boundedness Theorem, a convergent sequence must be bounded. It turns out to be almost equivalent statement!

Theorem 1.1 (Bolzano-Weiestrass Theorem)**.** *Suppose $\{x_n\}_{n=1}^\infty$ is a bounded sequence, then it admits a convergent sub-sequence.*

As a application,

Corollary 1.1. *If $\{x_n\}_{n=1}^\infty$ is bounded such that all convergent subsequence has the same limit, then $\{x_n\}_{n=1}^\infty$ is convergent with the same limt.*

2. Limit Superior and Limit Inferior

remark: I am not following the approach in textbook.

Recall that we only concern the behaviour when $n \to +\infty$. The convergence is equivalent to say that x_n is stabilized somewhere. To capture the "stability", it is often useful to consider the Oscillation of the tails.

Definition 2.1. *Given a bounded sequence $\{x_n\}_{n=1}^\infty$. Define*

(1)
$$\limsup_{n\to+\infty} x_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} x_n = \lim_{k\to+\infty} \sup_{n\geq k} x_n;$$

(2)
$$\liminf_{n\to+\infty} x_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} x_n = \lim_{k\to+\infty} \inf_{n\geq k} x_n.$$

Here the limits **Always** *exist by monotone convergence theorem. (1) capture the "max" of tail while (2) capture the "min".*

We have the equivalent form of definition (also equivalent to the one from the textbook).

Theorem 2.1. *Given a bounded sequence $\{x_n\}_{n=1}^\infty$, the followings are equivalent.*

1. $x = \limsup_{n\to+\infty} x_n$;
2. *For $\varepsilon > 0$, there are at most finitely many n such that $x+\varepsilon < x_n$ but infinity many n so that $x - \varepsilon < x_n$;*
3. $x = \inf V$ *where* $V = \{v \in \mathbb{R} : v < x_n$ *for at most finitely many* $n\}$;
4. $x = \sup S$ *where* $S = \{s \in \mathbb{R} : s = \lim_{k\to+\infty} x_{n_k}$ *for some* $\{n_k\}_{k=1}^\infty\}$.

Proof. (1) $\Rightarrow$ (2):

For all $\varepsilon > 0$, there is $k_0 \in \mathbb{N}$ such that for all $m \geq k > k_0$,

$$x + \varepsilon > \sup_{n \geq k} x_n \geq x_m.$$

Hence,

$$|\{i : x_i \geq x + \varepsilon\}| < +\infty$$

Moreover, $x - \varepsilon < \sup_{n \geq k} x_n$ for all $k \in \mathbb{N}$. Therefore, for each $k \in \mathbb{N}$, there is $n_k \geq k$ such that $x - \varepsilon < x_{n_k}$. Since $k \to +\infty$,

$$|\{i : x_i > x - \varepsilon\}| = +\infty.$$

(2) $\Rightarrow$ (3):

By (2), $x + \varepsilon \in V$ and hence $x + \varepsilon \geq \inf V$ for all $\varepsilon > 0$. By letting $\varepsilon \to 0$, we have

$$x \geq \inf V.$$

Suppose $x > \inf V$, there is $\varepsilon_0 > 0$ and $v \in V$ such that

$$x - \varepsilon_0 > v.$$

By (2) again, there are infinitely many x_n so that

$$x_n > x - \varepsilon_0 > v$$

which contradicts with $v \in V$. Hence $x = \inf V$.

(3) $\Rightarrow$ (4): We claim something slightly stronger: $\inf V = \sup S$.

Let $v \in V$, since there are at most finitely many x_n such that $v < x_n$. There is $N \in \mathbb{N}$ such that for all $n > N$, $v \geq x_n$. Let $s \in S$, there is n_k such that $x_{n_k} \to s$. Applying the properties of v on x_{n_k}, we have for all $k > N$,$v \geq x_{n_k}$. Hence,

$$v \geq s.$$

The inequality is true for all $s \in S, v \in V$. Hence, $\inf V \geq \sup S$.

We now claim that $\inf V = \sup S$. If not, there is $\varepsilon_0 > 0$ such that

$$a = \inf V - \varepsilon_0 > \sup S.$$

There is $N \in \mathbb{N}$ such that for all $n > N$, $a \geq x_n$. Since otherwise, we can find a subsequence x_{n_k} such that $a < x_{n_k}$ for all k. By Bolzano-Weiestrass Theorem, there is $x_{n_{k_j}}$ which converges to some $s \in S$ as $j \to +\infty$ so that $a \leq s \leq \sup S$ which is impossible. Therefore,

$$|\{n : a < x_n\}| < +\infty$$

which implies $a \in V$ and hence $a \geq \inf V = a + \varepsilon_0$. This is impossible.

(4) $\Rightarrow$ (1):

Let $s \in S$, there is $x_{n_k} \to s$. On the other hand, for all $k \in \mathbb{N}$,

$$\sup_{n\geq k} x_n \geq x_{n_k}.$$

By passing $k \to +\infty$, we have $\limsup_{n\to+\infty} x_n \geq s$ and hence

$$\limsup_{n\to+\infty} x_n \geq \sup S.$$

Denote $\bar{x} = \limsup_{n\to+\infty} x_n$. To show the opposite inequality, let $\varepsilon > 0$, we have for all $k \in \mathbb{N}$,

$$\bar{x} - \varepsilon < \sup_{n\geq k} x_n.$$

Therefore, for all $k \in \mathbb{N}$, there is x_{n_k} such that $\bar{x} - \varepsilon < x_{n_k}$. Using the construction of sub-sequence in previous lecture, we might assume $\{x_{n_k}\}$ forms a sub-sequence. By Bolzano-Weiestrass Theorem, there is $x_{n_{k_j}} \to s$ for some $s \in S$ as $j \to +\infty$. This shows

$$\bar{x} - \varepsilon \leq s \leq \sup S, \quad \forall \varepsilon > 0.$$

By letting $\varepsilon \to 0$, we have

$$\bar{x} \leq \sup S.$$

This completes the proof. □

The importance of lim sup and lim inf is that they always exist (without checking anything!!!!).

Theorem 2.2. *Given a bounded sequence $\{x_n\}$, it is convergent if and only if*

$$\limsup_{n\to+\infty} x_n = \liminf_{n\to+\infty} x_n.$$

Proof. Suppose the sequence is convergent: $x_n \to x$ for some $x \in \mathbb{R}$. For all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$|x_n - x| < e.$$

And hence,for all $k > N$,

$$x - \varepsilon \leq \inf_{n\geq k} x_n \leq \sup_{n\geq k} x_n \leq x + \varepsilon.$$

Let $k \to +\infty$ and followed by $\varepsilon \to 0$, we have

$$x \leq \liminf_{n\to+\infty} x_n \leq \limsup_{n\to+\infty} x_n \leq x.$$

To prove the opposite direction, let x be the common limit. Then for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $k > N$,

$$\sup_{n\geq k} x_n < x + \varepsilon, \quad \inf_{n\geq k} x_n > x - \varepsilon,$$

which shows that for all $n > N$,

$$x - \varepsilon < x_n < x + \varepsilon.$$

This completes the proof.

□